FDTD Modeling of Wave Propagation in Dispersive Media by Using the Mobius Transformation Technique

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***Abstract*—This paper introduces a technique for finite-difference time-domain modeling of wave propagation in general Mth-order dispersive media. Ohm's law in the Laplace domain with an Mth-order rational model for the complex conductivity is considered as a constitutive relation. In order to discretize this model, the complex conductivity is mapped onto the Z-transform domain by means of the Mobius transformation. This leads finally to a set of difference equations that is consistent with Yee's scheme. The resulting formulation is explicit, it has a second-order accuracy, and the need for additional storage variables is minimal. The numerical stability problem is discussed and the numerical dispersion equation for Mth-order media is given.**

***Index Terms*—Dispersive media, FDTD methods, Mobius transformation.**

I. Introduction

The applicability of the conventional finite-difference time-domain (FDTD)—Yee's scheme [1]—is restricted to linear, isotropic, nondispersive media. Of course, it is possible to handle media with frequency-dependent constitutive characteristics by dividing the frequency band of interest into a number of subbands for which the constitutive parameters are nearly constant and by performing one FDTD computer run for each subband. However, this is a cumbersome approach when different types of dispersive media are present and, furthermore, the broad-band capability of the FDTD method is lost.

Over the last decade, a number of researchers have realized that it is necessary to extend the conventional FDTD method to incorporate dispersive media. The currently available techniques for handling dispersive media with FDTD simulators are often grouped under three main categories: recursive convolution methods [2]–[6], auxiliary differential equation (ADE) techniques [7]–[12], and the Z-transform method [13]. A fourth technique has recently been introduced which basically consists of a direct fitting of the constitutive characteristics of the medium into a discrete-time frequency-domain conductivity or permittivity function [14], [15]. Several authors have carried out investigations of the numerical properties of these methods as well as comparative studies of their accuracy and computational costs [16]–[20].

This paper presents an alternative technique for FDTD modeling of wave propagation in Mth-order dispersive media. This technique basically consists of considering Ohm's law in the Laplace domain as a media constitutive relation assuming an Mth-order rational model for the complex conductivity. This relation in then expressed in the Z-transform domain by means of the Mobius transformation (MT). Finally, the resulting expression is written in difference form leading to a formulation that is consistent with Yee's scheme. We discuss the numerical stability of this technique and derive its numerical dispersion equation for a general Mth-order dispersive medium. Recently, this method has been successfully applied to the incorporation of linear lumped networks into FDTD simulators [21]. Throughout this paper, we will refer to this technique as the MT technique.

To illustrate the validity of the proposed technique, two examples with exact solutions have been considered: the computation of the reflection and transmission coefficients of a dielectric slab characterized by three Debye poles ($M = 3$), and the computation of the reflection coefficient of a half-space medium characterized by two Lorentz pole-pairs ($M = 4$). In both cases, the results obtained by FDTD under broad-band excitation have been compared with the exact data.

II. Theory

A. The Model

The conventional FDTD method is based on the discretization of the time-dependent Maxwell's curl equations in linear, isotropic, nondispersive media. To incorporate a dispersive medium into FDTD simulators, first a current density term $\vec{J}$ is added to Maxwell–Ampere's equation

$$\nabla \times \vec{H}(\vec{r}, t) = \epsilon_\infty \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} + \vec{J}(\vec{r}, t) \tag{1}$$

where ϵ_∞ is the permittivity at infinite frequency. Then, Ohm's law in the Laplace domain is considered to be the medium's constitutive equation

$$\vec{J}(\vec{r}, s) = \sigma(\vec{r}, s)\vec{E}(\vec{r}, s) \tag{2}$$

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where $\sigma(\vec{r}, s)$ represents the Laplace domain conductivity. In the following, we will assume $\sigma(\vec{r}, s)$ to be a rational function of the complex frequency s as

$$\sigma(\vec{r}, s) = \frac{\sum_{m=0}^{M} a_m(\vec{r}) s^m}{\sum_{m=0}^{M} b_m(\vec{r}) s^m} \tag{3}$$

where a_m and b_m are real-value coefficients and M is an integer denoting the order of the medium.

B. Discretization

Equation (1) is discretized by using Yee's scheme with a time average for the current density term

$$E_\alpha^{n+1}(\vec{r}_{E_\alpha}) = E_\alpha^n(\vec{r}_{E_\alpha}) + \frac{\Delta_t}{\epsilon_\infty}\left\{\left[\nabla \times \vec{H}\right]_\alpha^{n+(1/2)}(\vec{r}_{E_\alpha}) - \tfrac{1}{2}\left[J_\alpha^n(\vec{r}_{E_\alpha}) + J_\alpha^{n+1}(\vec{r}_{E_\alpha})\right]\right\} \tag{4}$$

where $\alpha = x, y, z$ and $\vec{r}_{E_\alpha}$ denotes the spatial position of E_α at Yee's cell. In (4), the curl term is written in compact form; detailed expressions for this term can be found in [1].

For the sake of brevity, in the following we will not specify the spatial point $\vec{r}_{E_\alpha}$ at which the equations are evaluated, because this point is the same for all the remaining difference equations.

Ohm's law is discretized by directly mapping the Laplace domain onto the Z-transform domain. To this end, the MT[1]

$$s = \frac{2}{\Delta_t}\frac{1 - Z^{-1}}{1 + Z^{-1}} \tag{5}$$

is applied to (2), resulting in

$$\vec{J}(\vec{r}, Z) = \sigma(\vec{r}, Z)\vec{E}(\vec{r}, Z) \tag{6}$$

with $\sigma(\vec{r}, Z)$, which represents the Z-transform domain conductivity, given by

$$\sigma(\vec{r}, Z) = \frac{N(\vec{r}, Z)}{D(\vec{r}, Z)} = \frac{1 + \sum_{m=1}^{M} c_m(\vec{r}) Z^{-m}}{\sum_{m=0}^{M} d_m(\vec{r}) Z^{-m}} \tag{7}$$

where the coefficients c_m and d_m are related to a_m and b_m and to the time step Δ_t. Expressions to compute these coefficients are given in the Appendix.

Equation (6) can now be written in difference form by simply considering the relationship $Z^{-m}F(Z) \leftrightarrow F^{n-m}$, which leads to

$$E_\alpha^{n+1} + \sum_{m=1}^{M} c_m E_\alpha^{n-m+1} = \sum_{m=0}^{M} d_m J_\alpha^{n-m+1}. \tag{8}$$

[1]This conformal transformation is also known as a "bilinear" or "linear-fractional" transformation.

This high-order difference equation is coupled to (4) since both equations involve the quantities E_α^{n+1} and J_α^{n+1}. To maintain the explicit nature of the conventional FDTD method, (4) and (8) must be solved for E_α^{n+1} and J_α^{n+1} before encoding them into an FDTD simulator. This results in

$$E_\alpha^{n+1} = \frac{1}{\frac{d_0\epsilon_\infty}{\Delta_t} + \frac{1}{2}} \cdot \left\{\frac{d_0\epsilon_\infty}{\Delta_t} E_\alpha^n + d_0\left[\nabla \times \vec{H}\right]_\alpha^{n+(1/2)} - \tfrac{1}{2} A_\alpha + \frac{d_1 - d_0}{2} J_\alpha^n\right\} \tag{9a}$$

$$J_\alpha^{n+1} = \frac{1}{d_0}\left(E_\alpha^{n+1} + A_\alpha\right) \tag{9b}$$

where

$$A_\alpha = \sum_{m=1}^{M}\left(c_m E_\alpha^{n-m+1} - d_m J_\alpha^{n-m+1}\right).$$

The above equations allow Mth-order dispersive media to be incorporated into FDTD simulators. With respect to the conventional FDTD method, the implementation of (9a) and (9b) requires an additional computational cost of $2M - 1$ storage variables and $2M + 4$ real products per each electric field component and per cell.

C. Efficient (Minimal) Implementation

To reduce memory requirements, more efficient implementations of (8) can be devised. Such an equation, in fact, can be interpreted as an infinite-impulse response (IIR) digital filter. As in [14], we adopt the transposed direct form II to implement (8). This is a filter realization that minimizes memory requirements [22]. Following this approach, (8) is expressed in the following equivalent form:

$$\begin{aligned} E_\alpha^{n+1} &= W_{\alpha,1}^n + d_0 J_\alpha^{n+1} \\ W_{\alpha,m}^{n+1} &= W_{\alpha,m+1}^n - c_m E_\alpha^{n+1} + d_m J_\alpha^{n+1} \\ W_{\alpha,M}^{n+1} &= d_M J_\alpha^{n+1} - c_M E_\alpha^{n+1} \end{aligned} \tag{10}$$

where $m = 1, 2, \ldots, M-1$, and $W_{\alpha,m}$ are auxiliary variables. This set of $M+2$ first-order difference equations is, again, coupled to (4). For comparison with previously published methods, we will separately discuss the decoupling of these equations for the particular cases $M = 0$ and $M = 1$ and for the general case $M > 1$.

1) Zeroth-Order Media: For zeroth-order media, the conductivity is static, thus

$$\sigma(s) = \sigma(Z) = \sigma_0.$$

In this case, (4) and (10) reduce directly to

$$E_\alpha^{n+1} = \frac{2\Delta_t}{2\epsilon_\infty + \sigma_0\Delta_t}\left[\nabla \times \vec{H}\right]_\alpha^{n+(1/2)} + \frac{2\epsilon_\infty - \sigma_0\Delta_t}{2\epsilon_\infty + \sigma_0\Delta_t} E_\alpha^n$$

which is seen to be the equation obtained to update the E_α component of the electric field when the time-average scheme is adopted to incorporate lossy media into FDTD simulators [23].

2) First-Order Media: First-order media, such as cold plasma and Debye media, are characterized by a conductivity function of the form

$$\sigma(s) = \frac{a_0 + a_1 s}{b_0 + b_1 s}.$$

For cold plasma, the coefficients of $\sigma(s)$ are $a_0 = \epsilon_0 \omega_p^2$, $a_1 = 0$, $b_0 = \omega_c$, and $b_1 = 1$, where ω_p and ω_c are the plasma and the collision angular frequencies, respectively. For Debye media, the coefficients of $\sigma(s)$ are $a_0 = 0$, $a_1 = \epsilon_s - \epsilon_\infty$, $b_0 = 1$, and $b_1 = \tau$, where ϵ_s is the static permittivity and τ is the relaxation time constant.

For these media, by decoupling (4) and (10) and eliminating $W_{\alpha,1}$, we obtain

$$E_\alpha^{n+1} = \frac{1}{\frac{\epsilon_\infty}{\Delta_t} + \frac{1}{2d_0}} \cdot \left\{ \left[\nabla \times \vec{H}\right]_\alpha^{n+(1/2)} + \left(\frac{\epsilon_\infty}{\Delta_t} - \frac{c_1}{2d_0}\right) E_\alpha^n + \frac{d_1 - d_0}{2d_0} J_\alpha^n \right\} \tag{11a}$$

$$J_\alpha^{n+1} = \frac{1}{d_0}\left(c_1 E_\alpha^n + E_\alpha^{n+1} - d_1 J_\alpha^n\right). \tag{11b}$$

For cold plasma, the equations in (11) are identical to the equations of the scheme called the "New DI method" given in [19]. Also, the equations in (11) are the same as the equations given in [7] (this can be shown by simply letting $\omega_0 = 0$ and eliminating the polarization vector in [7]).

3) Mth-Order Media ($M > 1$): Decoupling, again, (4) and (10), and eliminating $W_{\alpha,M}$, we obtain

$$E_\alpha^{n+1} = \frac{1}{\frac{\epsilon_\infty}{\Delta_t} + \frac{1}{2d_0}} \cdot \left\{ \frac{\epsilon_\infty}{\Delta_t} E_\alpha^n + \left[\nabla \times \vec{H}\right]_\alpha^{n+(1/2)} + \frac{1}{2d_0} W_{\alpha,1}^n - \frac{1}{2} J_\alpha^n \right\}$$

$$J_\alpha^{n+1} = \frac{1}{d_0}\left(E_\alpha^{n+1} - W_{\alpha,1}^n\right)$$

$$W_{\alpha,m}^{n+1} = W_{\alpha,m+1}^n - c_m E_\alpha^{n+1} + d_m J_\alpha^{n+1}$$

$$W_{\alpha,M-1}^{n+1} = -c_M E_\alpha^n - c_{M-1} E_\alpha^{n+1} + d_M J_\alpha^n + d_{M-1} J_\alpha^{n+1}$$

where $m = 1, 2, \ldots, M-2$. With respect to the conventional FDTD method, the implementation of these equations requires $2M + 4$ additional real products and only M additional storage variables (one for J_α and $M - 1$ for the auxiliary variables $W_{\alpha,m}$).

D. A Time-Domain Interpretation of the MT

The application of the MT, as introduced in the above sections, is similar to its use in designing digital filters, as can be seen in the signal processing literature (see, for example, [22]). In that context, the application of the MT is usually interpreted to be analogous to an integration in the time domain by means of the trapezoidal rule. An alternative interpretation of the time-domain discretization scheme underlying the application of (5) to (2) is developed in this section.

Equation (8) can be rewritten in terms of the coefficients a_m and b_m and in operational form as

$$\sum_{m=0}^{M} \frac{a_m}{\Delta_t^m} \boldsymbol{\delta}_t^m \boldsymbol{\mu}_t^{M-m} \vec{E}^n = \sum_{m=0}^{M} \frac{b_m}{\Delta_t^m} \boldsymbol{\delta}_t^m \boldsymbol{\mu}_t^{M-m} \vec{J}^n \tag{12}$$

where $\boldsymbol{\delta}_t$ and $\boldsymbol{\mu}_t$ are the central difference and the average operators, respectively. Their definitions can be found in [24].

On the other hand, considering the relationship $s^m F(s) \leftrightarrow d_t^m F(t)$, where d_t denotes derivation with respect to time, (2) can be expressed in the continuous-time domain as the following high-order ordinary differential equation:

$$\sum_{m=0}^{M} a_m d_t^m \vec{E}(t) = \sum_{m=0}^{M} b_m d_t^m \vec{J}(t). \tag{13}$$

Comparing (12) with (13), we find that the application of (5) to (2) is equivalent to the discretization of (13) by using a difference scheme consisting of the following second-order approximation:

$$d_t^m F(t) \simeq \frac{1}{\Delta_t^m} \boldsymbol{\delta}_t^m \boldsymbol{\mu}_t^{M-m} F^n$$

for the m-times derivative operator. Note that this approximation applies also for $m = 0$.

III. Numerical Stability and Dispersion Analysis

A. Numerical Stability

To study the numerical stability of the proposed scheme, the von Neumann technique is adopted. This method, which basically consists of expressing the relevant difference equations in the spectral domain, leads to a stability polynomial $S(Z)$. The condition for stability is that all the roots Z_i of $S(Z)$ will be inside, or on, the unit circle in the Z-transform domain, i.e., $|Z_i| \le 1$.

For simplicity, instead of working directly with Maxwell's equations, the time-dependent wave equation for the electric field in a homogeneous dispersive medium is considered

$$\nabla^2 \vec{E}(t) - \mu\epsilon_\infty \partial_t^2 \vec{E}(t) - \mu \partial_t \vec{J}(t) = 0$$

where ∂_t denotes partial derivation with respect to time. According to the conventional FDTD method, this equation is approximated by

$$\left(\sum_{\alpha=x,y,z} \frac{\boldsymbol{\delta}_\alpha^2}{\Delta_\alpha^2} - \mu\epsilon_\infty \frac{\boldsymbol{\delta}_t^2}{\Delta_t^2}\right) \vec{E}^n - \mu \frac{\boldsymbol{\delta}_t \boldsymbol{\mu}_t}{\Delta_t} \vec{J}^n = 0 \tag{14}$$

where $\boldsymbol{\delta}_\alpha$ denotes the central difference operator with respect to the coordinate indicated by the subscript.

Starting from (12) and (14), and following the procedure described in [24], we arrive at the stability polynomial

$$S(Z) = D(Z)[4\nu^2 Z + (Z-1)^2] + \frac{\Delta_t}{2\epsilon_\infty} N(Z)(Z^2 - 1) \tag{15}$$

where, for a general Mth-order medium, $N(Z)$ and $D(Z)$ are, respectively, the numerator and denominator of $\sigma(Z)$ [given in (7)], and

$$\nu^2 = (c_\infty \Delta_t)^2 \sum_{\alpha=x,y,z} \frac{\sin^2\left(\frac{\tilde{k}_\alpha \Delta_\alpha}{2}\right)}{\Delta_\alpha^2}$$

where $\tilde{k}_\alpha$ and Δ_α are, respectively, the numerical wavenumber and the size of the discretization cell in the α-direction, and $c_\infty = 1/\sqrt{\mu\epsilon_\infty}$.

The number of Z_i roots of $S(Z)$ is $M+2$ and their location depends on the particular medium under consideration.

1) Zeroth-Order Media: For zeroth-order media $(\sigma(Z) = \sigma_0)$, the resulting polynomial has been previously studied [25, Sec. III.A] showing that the stability limit of the conventional FDTD method is preserved, provided that $\sigma_0 \geq 0$.

2) First-Order Media: For cold plasma, the stability polynomial is

$$\begin{aligned} S_{\mathrm{P}}(Z) = &(2\overline{\omega}_c + \overline{\omega}_p^2 + 4)Z^3 \\ &+ [8\nu^2(\overline{\omega}_c + 2) - 2\overline{\omega}_c + \overline{\omega}_p^2 - 12]Z^2 \\ &+ [8\nu^2(\overline{\omega}_c - 2) - 2\overline{\omega}_c - \overline{\omega}_p^2 + 12]Z \\ &+ 2\overline{\omega}_c - \overline{\omega}_p^2 - 4 \end{aligned}$$

with $\overline{\omega}_p = \omega_p \Delta_t$ and $\overline{\omega}_c = \omega_c \Delta_t$. Following the procedure given in [24], it is easy to show that the stability limit of the conventional FDTD method is preserved provided $\omega_c \geq 0$.

For Debye media, the stability polynomial is the same as the stability polynomial given in [24, eq. (11)] for the scheme introduced in [8]. Thus, for Debye media, the MT technique is equivalent to the scheme given in [8].

3) Mth-Order Media ($M > 1$): The higher the order of the medium, the greater the difficulty in obtaining closed-form stability expressions. This difficulty can be alleviated with the technique described in [24]. As an example, we consider a Lorentz medium with a conductivity

$$\sigma(s) = \frac{(\epsilon_s - \epsilon_\infty)\omega_0^2 s}{\omega_0^2 + 2\delta_0 s + s^2}$$

where ω_0 is the resonant angular frequency and δ_0 is the damping coefficient. For this case, the stability polynomial is

$$\begin{aligned} S_{\mathrm{L}_1}(Z) = &(\overline{\omega}_0^2\overline{\epsilon}_s + 4\overline{\delta}_0 + 4)Z^4 \\ &+ 4[\nu^2(\overline{\omega}_0^2 + 4\overline{\delta}_0 + 4) - 2\overline{\delta}_0 - 4]Z^3 \\ &+ 2[4\nu^2(\overline{\omega}_0^2 - 4) - \overline{\omega}_0^2\overline{\epsilon}_s + 12]Z^2 \\ &+ 4[\nu^2(\overline{\omega}_0^2 - 4\overline{\delta}_0 + 4) + 2\overline{\delta}_0 - 4]Z \\ &+ \overline{\omega}_0^2\overline{\epsilon}_s - 4\overline{\delta}_0 + 4 \end{aligned}$$

where $\overline{\omega}_0 = \omega_0\Delta_t$, $\overline{\delta}_0 = \delta_0\Delta_t$, and $\overline{\epsilon}_s = \epsilon_s/\epsilon_\infty$. This polynomial is the same as the stability polynomial obtained in [24] for the scheme called a "New difference scheme for Lorentz media." In fact, that scheme and the MT technique for Lorentz media have the same numerical properties. Therefore, as was shown in [24], the MT technique for Lorentz media preserves the stability limit of the conventional FDTD method.

B. Numerical Dispersion Equation

The numerical dispersion equation can be easily obtained by simply evaluating $S(Z)$ on the unit circle in the Z-transform domain and equating the result to zero, thus by letting $S(e^{\mathbf{j}\omega\Delta_t}) = 0$.

For an Mth-order dispersive medium, the numerical dispersion equation reads

$$\frac{\sin^2\left(\frac{\omega\Delta_t}{2}\right)}{\Delta_t^2}\mu\left[\epsilon_\infty - \frac{\mathbf{j}\tilde{\sigma}}{\frac{2}{\Delta_t}\tan\left(\frac{\omega\Delta_t}{2}\right)}\right] = \sum_{\alpha=x,y,z}\frac{\sin^2\left(\frac{\tilde{k}_\alpha\Delta_\alpha}{2}\right)}{\Delta_\alpha^2} \tag{16}$$

where

$$\tilde{\sigma} \equiv \sigma(e^{\mathbf{j}\omega\Delta_t}) = \frac{\sum_{m=0}^{M} a_m\left[\frac{\mathbf{j}2}{\Delta_t}\tan\left(\frac{\omega\Delta_t}{2}\right)\right]^m}{\sum_{m=0}^{M} b_m\left[\frac{\mathbf{j}2}{\Delta_t}\tan\left(\frac{\omega\Delta_t}{2}\right)\right]^m} \tag{17}$$

is the discrete-time frequency-domain conductivity, usually called numerical conductivity in the FDTD literature. Comparing the above expression with the continuous-time frequency-domain conductivity [(3) evaluated at $s = \mathbf{j}\omega$], it can be observed that the discretization of (3) by the MT involves a mapping

$$\omega \to \frac{2}{\Delta_t}\frac{\sin(\omega\Delta_t/2)}{\cos(\omega\Delta_t/2)} \tag{18}$$

from the continuous- onto the discrete-time frequency domains. As consistency demands, $\tilde{\sigma}$ tends to $\sigma(s)|_{s=\mathbf{j}\omega}$ as $\omega\Delta_t$ tends to zero.

For zeroth-order media, (16) leads to the numerical dispersion equation given in [25].

For cold plasma, the resulting numerical dispersion equation is the same as that given in [19, eq. (9)] for the scheme labeled "New DI method," and for Debye media is the same as [16, eq. (3)].

As examples of high-order media, we consider media characterized by M Debye poles or by M Lorentz pole-pairs. For the former, the numerical conductivity is given by

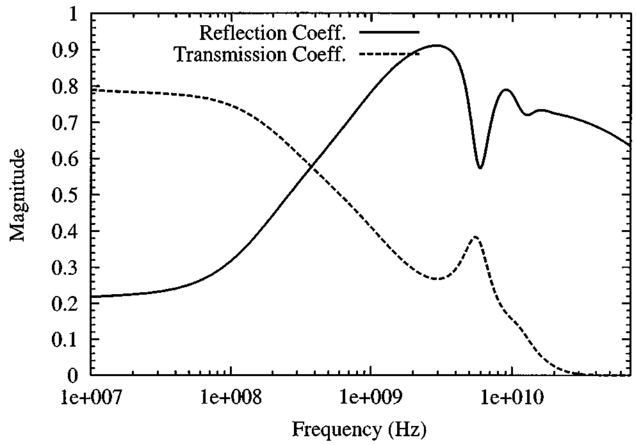
$$\tilde{\sigma}_{\mathrm{D}_M} = \sum_{m=1}^{M}\frac{\Delta\epsilon_m\frac{\mathbf{j}2}{\Delta_t}\tan\left(\frac{\omega\Delta_t}{2}\right)}{1 + \tau_m\frac{\mathbf{j}2}{\Delta_t}\tan\left(\frac{\omega\Delta_t}{2}\right)}$$

with $\Delta\epsilon_m = \epsilon_{s_m} - \epsilon_\infty$, and for the latter

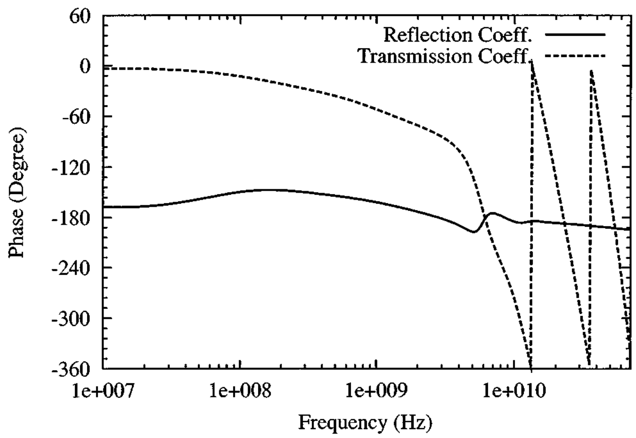
$$\tilde{\sigma}_{\mathrm{L}_M} = \sum_{m=1}^{M}\frac{\frac{\mathbf{j}2}{\Delta_t}(\epsilon_s - \epsilon_\infty)G_m\omega_m^2\tan\left(\frac{\omega\Delta_t}{2}\right)}{\omega_m^2 + 2\delta_m\frac{\mathbf{j}2}{\Delta_t}\tan\left(\frac{\omega\Delta_t}{2}\right) - \frac{4}{\Delta_t^2}\tan^2\left(\frac{\omega\Delta_t}{2}\right)}$$

with $\sum_{m=1}^{M} G_m = 1$.

It is known that Yee's scheme involves the transformation $\omega \to (2/\Delta_t)\sin(\omega\Delta_t/2)$ from the continuous- onto the discrete-time frequency domains. We can then associate the cosine term in (18) to the characteristic time constants of the media (i.e., τ_m, ω_m^{-1}, and δ_m^{-1}.). This line of reasoning leads to the

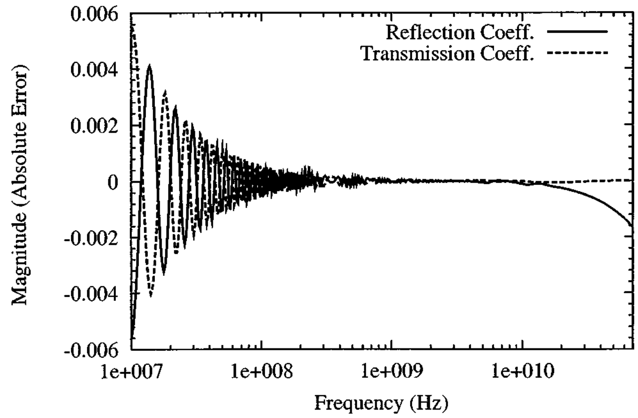


(a)



(b)

Fig. 1. Exact reflection and transmission coefficients for a slab characterized by three Debye poles. (a) Magnitude. (b) Phase.



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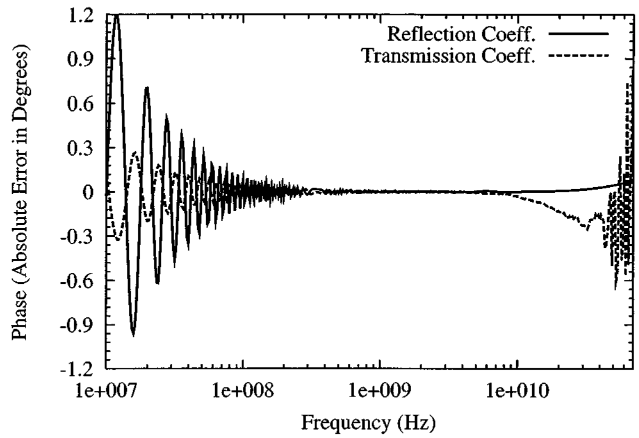

(b)

Fig. 2. Absolute error of the reflection and transmission coefficients for the example of Fig. 1. (a) Magnitude. (b) Phase.

definition of numerical values for these constants [16]. It can be shown from (16) that the MT technique produces numerical constants of the form

$$\tilde{\tau}_m = \tau_m / \cos(\omega \Delta_t / 2)$$

for all the characteristic time constants of the medium. The cosine terms arise from the averaging process involved in the MT.

IV. Validation

In order to validate the MT method presented above, we have considered two examples with exact solutions.

First we computed the reflection and transmission coefficient of a dielectric slab with three Debye poles and a static conductivity. The parameters of this medium were $\sigma_0 = 0.106$ S/m, $\epsilon_\infty = 4.3\epsilon_0$, $\tau_1 = (5.2\pi)^{-1}$ μs, $\tau_2 = (680\pi)^{-1}$ μs, $\tau_3 = (46\pi)^{-1}$ ns, $\Delta\epsilon_1 = 1970\epsilon_0$, $\Delta\epsilon_2 = 30.8\epsilon_0$, and $\Delta\epsilon_3 = 41.3\epsilon_0$ [12]. The size of the spatial cell taken to run this example was $\Delta_z = 37.5$ μm and the time step $\Delta_t = \Delta_z / c \simeq 0.125$ ps. The width of the slab was $d = 100\Delta_z$. Fig. 1(a) shows the exact solution for the magnitude of the reflection and transmission coefficients. The exact solution for the phases is depicted in Fig. 1(b). Fig. 2(a) shows the absolute error of the magnitudes of the reflection and transmission coefficients obtained by the MT method as compared to the exact solution. The absolute error for the phases is shown in Fig. 2(b). All these results have been plotted over a frequency band that runs from 10 MHz to 70 GHz. The ripple in the curves of absolute error, observed in the lower part of the frequency band (approximately until 300 MHz), is due to the truncation of the time-domain response. To obtain these results, the FDTD simulation was run for 1 million time steps! Apart from ripple effects, it can be observed that the absolute errors increase with frequency.

As a second example, we considered a medium with two Lorentz pole-pairs ($M = 4$). The parameters were $\epsilon_s = 3\epsilon_0$, $\epsilon_\infty = 1.5\epsilon_0$, $\omega_1 = 2\pi \times 20$ GHz, $\delta_1 = 0.1\omega_1$, $G_1 = 0.4$, $\omega_2 = 2\pi \times 50$ GHz, $\delta_2 = 0.1\omega_2$, and $G_2 = 0.6$ [12], [2]. The size of the spatial cell was $\Delta_z = 37.5$ μm and the time step $\Delta_t = \Delta_z / c \simeq 0.125$ ps. Proceeding analogously to the previous example, first we obtained the exact solution for the magnitude and phase of the reflection coefficient. These results are shown in Fig. 3(a) and (b), respectively. The absolute errors of the magnitude and phase of the reflection coefficient are depicted in Fig. 4. In this case, the time-domain response is much shorter than in the previous example. As consequence, there is no significant ripple masking the behavior of the error at low frequencies. In this example, local maxima of the error curves are observed in the vicinity of the resonant frequencies of the medium.

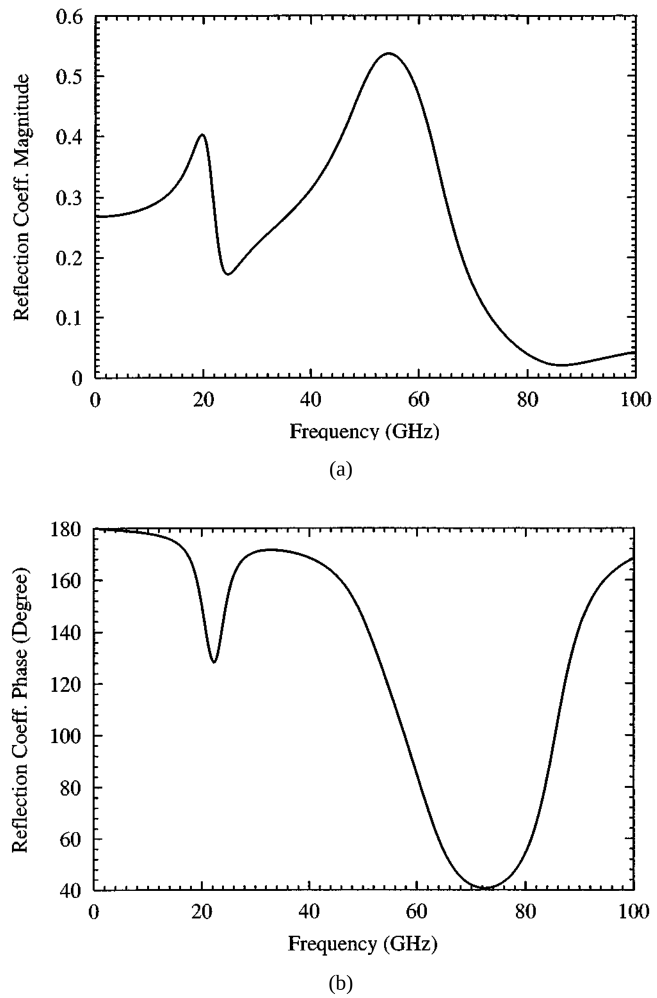

Fig. 3. Exact reflection coefficient for a half-space characterized by two Lorentz pole-pairs. (a) Magnitude. (b) Phase.

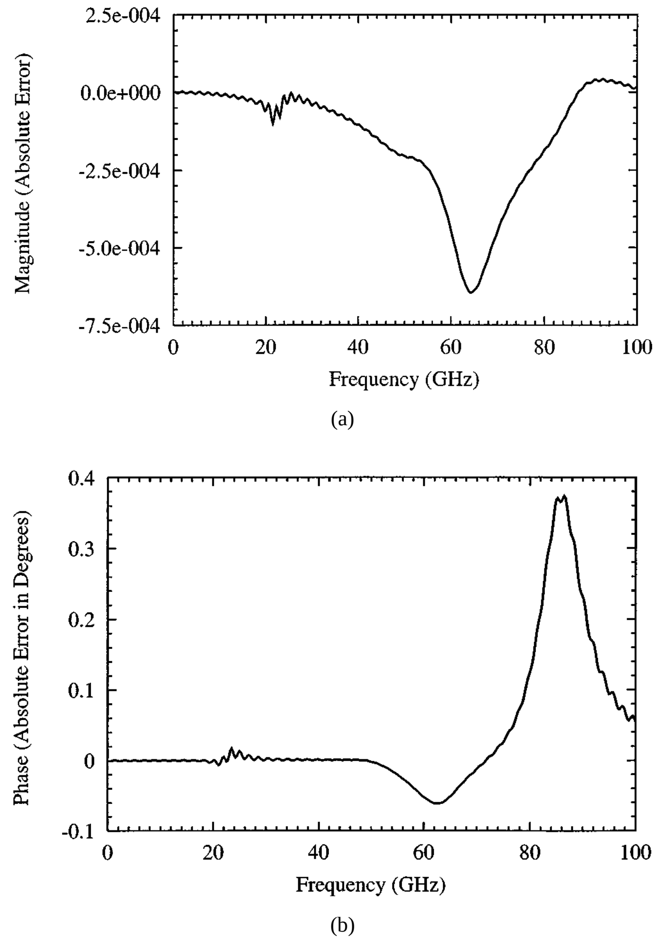

Fig. 4. Absolute error of the reflection coefficient for the example of Fig. 3. (a) Magnitude (b) Phase.

V. Conclusion

This paper has introduced the MT technique for FDTD modeling of wave propagation in general Mth-order dispersive media. Starting from a rational model of the conductivity in the Laplace domain and applying the MT, this technique leads to a set of difference equations. The resulting finite-difference scheme preserves the second-order accuracy and the explicit nature of the conventional FDTD method. Furthermore, the simulation of an Mth-order dispersive medium requires only M additional storage variables (per electric field component and per cell). The numerical stability of the MT technique has been analyzed by means of the von Neumann method and the numerical dispersion equation has also been derived. The validity of this new formulation has been shown for various high-order dispersive media.

When classifying the MT method into one of the categories mentioned in the introduction, it may appear at first sight that "this is a slightly different twist on the use of Z transforms." However, as has been shown in the paper, the MT method can be interpreted as an approximation of high-order ODEs by using centered finite differences. In this sense, the MT method is much closer to ADE techniques than to the Z transform method. In fact, for low-order media (plasmas, single Debye pole and single Lorentz pole-pair media) we have shown that the MT method is equivalent to previously reported ADE techniques. More precisely, the MT method provides a systematic formulation for general Mth-order media that can be interpreted as an ADE technique. The significant difference with respect to ADE techniques is that instead of dealing with high-order ODEs in the time domain, the MT method treats the problem in a much simpler way by first setting-up an equivalent problem in the Laplace domain and then going to the discrete-world by performing a MT to the Z domain. Finally, given the underlying equivalence of this method to ADE techniques, the MT method should also be applicable to nonlinear media as ADE techniques are.

Appendix

The coefficients c_m and d_m of (7) are given by

$$c_m = \frac{1}{c_0} \sum_{l=0}^{M} \frac{a_l}{2^{M-l}\Delta_t^l} \sum_{i=i_{\min}}^{i_{\max}} (-1)^i \binom{l}{i} \binom{M-l}{m-i}$$

with $m = 1, \ldots, M$ and

$$d_m = \frac{1}{c_0} \sum_{l=0}^{M} \frac{b_l}{2^{M-l}\Delta_t^l} \sum_{i=i_{\min}}^{i_{\max}} (-1)^i \binom{l}{i} \binom{M-l}{m-i}$$

with $m = 0, \ldots, M$. In the above expressions, $i_{\min} = \max(0, m + l - M)$, $i_{\max} = \min(m, l)$ and c_0 is a normalization constant given by

$$c_0 = \sum_{l=0}^{M} \frac{a_l}{2^{M-l} \Delta_t^l}.$$

The binomial coefficient is defined by

$$\binom{l}{i} = \frac{l!}{i!(l-i)!}.$$

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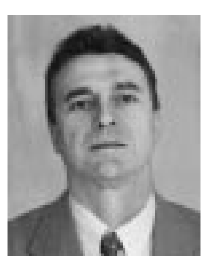

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